International Journal of Theoretical Physics, Vol. 32, No. 5, 1993

Calculating Wadati–Deguchi–Akutsu $N=3$ Knot Polynomials

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Received March 15, 1992

A method for calculating Wadati *et al.* $N=3$ knot polynomials is given. Recurrence relations are derived. Examples are given to illustrate the method. The method can be implemented on a computer to calculate the polynomials for complicated knots. An example is given using Mathematica.

1. INTRODUCTION

Knots play an important role in many fields of mathematics, physics, biology, etc. A general theory has been presented (Wadati *et al.*, 1989) to construct link polynomials, topological invariants for knots and links, from exactly solvable models.

Exact solvability of a model in statistical mechanics means that it is possible to evaluate physical quantities such as free energy and the one-point function (magnetization, density, etc.). The two-dimensional Ising model with nearest neighbor interactions is exactly solvable (Onsager, 1944).

The soliton is a nonlinear wave with the particle property and satisfies a nonlinear evolution equation. Using the inverse scattering method (Gardner *et al.*, 1967; Wadati and Toda, 1972), which is an extension of the Fourier transformation, the classical soliton system has been shown to be a completely integrable system (Zakharov and Shabat, 1972; Flaschka and Newell, 1975). The mapping of a field variable into the scattering data is a canonical transformation and in the scattering data space we can choose action–angle variables. Applying the inverse scattering method to quantum systems is called quantum inverse scattering (Faddeev, 1980; Wadati *et al.*, 1986). The quantum inverse scattering method provides a unified framework

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0020-7748/93/0500-0713$07.00/0

and a powerful method for studying solvable models. Satisfying the Yang–Baxter relation (Baxter, 1972) is a sufficient condition for solvability. An infinite number of solvable lattice models have been discovered (Wadati and Akutsu, 1988). Akutsu *et al.* (1989) showed that exactly solvable models in statistical mechanics are extremely informative about the classification problems of configurations of strings. A string is defined as a very long, very thin object which in physics could be a vortex line, magnetic flux, a dislocation, a particle trajectory, etc. Particles in high-energy physics are considered as string vibrations.

One should also note the significant role of knots in DNA research. DNA molecules are long and stringlike, and can naturally take a closed circular form. The knots which arise in this field are complicated (Sumners, 1987).

Jones polynomials play an important role in the theory of knots and the above subjects, but they cannot distinguish between some knots, e.g., the Birman example (Birman, 1985). The Wadati *et al.* $N=3$ polynomials do not have this problem, since, as shown in the Appendix, they can distinguish between the two knots of the Birman example.

In Section 2 we derive a recurrence relation to calculate the two-variable Jones polynomials and then a closed form method to calculate the same polynomials. We need to consider the case $N=2$ (Hecke algebra and Jones polynomials) because, according to a general theory presented by Wadati *et al.* (1989), to construct link polynomials the operators G_i are constructed from a composite string combining $N-1$ strings—generators g_i of the Hecke algebra—and attaching a projector $P_i^{(N)}$ at each end. Then we consider the case $N=3$ and derive a recurrence method to calculate the Wadati polynomials. We concentrate on the case $N=3$ because one can proceed in the same manner for $N>3$. To complete the formulation of the method, we give two relations which can be proved for any braid group. These two relations enable us to simplify the calculations for the trace function for complicated knots, as illustrated by examples in Section 3.

In Section 3 we present examples to illustrate our method. In the Appendix we present the Birman example calculations for $N=3$ knots, where we use Mathematica.

2. BASIC FORMULATION

The Hecke algebra $H(t, n)$ is an algebra generated by operators $g_1, g_2, \ldots, g_{n-1}$ satisfying

$$g_i g_k = g_k g_i, \qquad |i-k| \geq 2 \tag{2.1a}$$

$$g_{i+1} g_i g_{i+1} = g_i g_{i+1} g_i \tag{2.1b}$$

$$g_i^2 = (1-t) g_i + t \tag{2.2}$$

Let us write g_i as

$$g_i = \boldsymbol{m}_1(t, 1)g_i + \boldsymbol{m}_2(t, 1), \qquad \boldsymbol{m}_1(t, 1) = 1; \quad \boldsymbol{m}_2(t, 1) = 0$$

On the other hand, from equation (2.2),

$$g_i^2 = \boldsymbol{m}_1(t, 2)g_i + \boldsymbol{m}_2(t, 2), \qquad \boldsymbol{m}_1(t, 1) = (1-t); \quad \boldsymbol{m}_2(t, 1) = t$$

Assume that

$$g_i^n = \boldsymbol{m}_1(t, n)g_i + \boldsymbol{m}_2(t, n) \tag{2.3}$$

Apply the operator g_i on both sides of equation (2.3) and use (2.2) to substitute for g_i^2 to get

$$g_i^{n+1} = \boldsymbol{m}_1(t, n+1)g_i + \boldsymbol{m}_2(t, n+1)$$

where

$$\boldsymbol{m}_1(t, n+1) = (1-t)\boldsymbol{m}_1(t, n) + \boldsymbol{m}_2(t, n)$$

$$\boldsymbol{m}_2(t, n+1) = t\boldsymbol{m}_1(t, n)$$

Hence, one can express g_i^n $(n > 0)$ in terms of g_i as given in equation (2.3) and the functions $\boldsymbol{m}_1(t, n)$ and $\boldsymbol{m}_2(t, n)$ can be calculated from the following recurrence relations:

$$\boldsymbol{m}_1(t, 1) = 1; \qquad \boldsymbol{m}_2(t, 1) = 0$$

$$\boldsymbol{m}_1(t, r) = (1-t)\boldsymbol{m}_1(t, r-1) + \boldsymbol{m}_2(t, r-1)$$

$$\boldsymbol{m}_2(t, r) = t\boldsymbol{m}_1(t, r-1), \qquad r \geq 2$$

If we calculate $\boldsymbol{m}_1(t, r)$, $\boldsymbol{m}_2(t, r)$ for $r = 1, 2, 3, 4$, we get

$$\boldsymbol{m}_1(t, 1) = 1; \qquad \boldsymbol{m}_2(t, 1) = 0$$

$$\boldsymbol{m}_1(t, 2) = 1 - t; \qquad \boldsymbol{m}_2(t, 2) = t$$

$$\boldsymbol{m}_1(t, 3) = 1 - t + t^2; \qquad \boldsymbol{m}_2(t, e) = t - t^2$$

$$\boldsymbol{m}_1(t, 4) = 1 - t + t^2 - t^3; \qquad \boldsymbol{m}_2(t, 4) = t - t^2 + t^3$$

From the above pattern one can derive the functions $\boldsymbol{m}_1(t, r)$ and $\boldsymbol{m}_2(t, r)$:

$$\boldsymbol{m}_1(t, r) = \sum_{k=0}^{k=r-1} (-1)^k t^k; \qquad \boldsymbol{m}_2(t, r) = 1 - \boldsymbol{m}_1(t, r); \qquad r \geq 1$$

So far we have considered the case $n > 0$; let us now deal with the case $n < 0$. Operating on both sides of equation (2.2) by g_i^{-1}, we get

$$g_i^{-1} = [g_i + (t-1)]/t \tag{2.4}$$

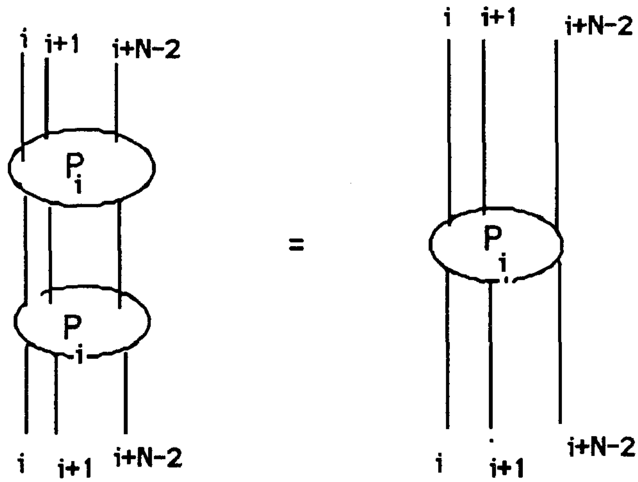

Fig. 1. A composite string. The two diagrams are equivalent since $P^2=P$.

or

$$g_i^{-1}=[v_1(t,1)g_i+v_2(t,1)]/t; \qquad v_1(t,1)=1, \quad v_2(t,1)=t-1$$

Set $n=-p$, and assume that

$$g_i^{-p}=[v_1(t,p)g_i+v_2(t,p)]/t^p \tag{2.5}$$

It is straightforward to show that

$$v_1(t,r)=\sum_{k=0}^{k=r-1}(-1)^{r-k-1}t^k; \qquad v_2(t,r)=t^r-\sum_{k=0}^{k=r-1}(-1)^{r-k-1}t^k, \qquad r\geq 1 \tag{2.6}$$

This completes the direct and recursive expressions for g_i^n in the Hecke algebra.

Starting from the generators $\{g_i\}$ of the Hecke algebra (Wadati *et al.*, 1989), composite braid operators $\{g_i\}$ are constructed using only the defining relations (2.1a), (2.1b) of the Hecke algebra operators $\{g_i\}$, from a composite string combining $N-1$ strings and attaching a projector $P_i^{(N)}$ at each end (Figure 1). It was shown (Wadati *et al.*, 1989) that the projector $P_i^{(N)}$ for $N=2, 3, 4, \ldots$, is derived through the recursion formula

$$P_i^{(N)}=P_i^{(N-1)}h_{i-N-3}^{(N)}P_i^{(N-1)}, \qquad P_i^{(2)}=1 \tag{2.7}$$

where

$$h_i^{(N)}=(\tau_{N-2}/\tau_{N-1})(t^{N-2}/\tau_{N-2}+g_i), \qquad \tau_m=1+t+\cdots+t^{m-1}$$

We write P_i instead of $P_i^{(N)}$ when no confusion arises.

The projector operator P_i defined by (2.7) satisfies the relations

(i) $$P_i^2 = P_i \tag{2.8a}$$

(ii) $$P_i \Delta_i^2 = P_i \tag{2.8b}$$

(iii) $$P_i(g_{i+N-2}g_{i+N-3}\ldots g_i) = (g_{i+N-2}g_{i+N-3}\ldots g_i)P_{i+1}$$
$$P_i(g_{i-N-2}^{-1}g_{i-N-3}^{-1}\ldots g_i^{-1}) = (g_{i-N-2}^{-1}g_{i-N-3}^{-1}\ldots g_i^{-1})P_{i+1} \tag{2.8c}$$

(iv) $$P_i g_k = P_i \quad \text{for} \quad k = i, i+1, \ldots, i+N-3 \tag{2.8d}$$

where the operator Δ_i is a half-twist,

$$\Delta_i = (g_i g_{i+1}\ldots g_{i+N-3})(g_i g_{i+1}\ldots g_{i+N-4})\ldots(g_i)$$

Denote the "spin-s" representation of B_n by $B_n^{[s]}$. Let $k = N-1 = 2s$ construct n sets of k strings and combine k strings into a composite string with projectors at both ends. The generator G_i is depicted in Figure 2. Describe an operator $G_i^{(N)}$ by

$$G_i^{(N)} = g_i^{(1)} g_i^{(2)} \ldots g_i^{(N-1)}$$

where

$$g_i^{(r)} = g_{ik+1-r}g_{ik+2-r}\ldots g_{k(i+1)-r}, \qquad r = 1, 2, \ldots, N-1$$

and the generator G_i of $B_n^{[s]}$ can be expressed as

$$G_i = P_{(i-1)k+1}^{(N)} P_{ik+1}^{(N)} G_i^{(N)} P_{(i-1)k+1}^{(N)} P_{ik+1}^{(N)}$$

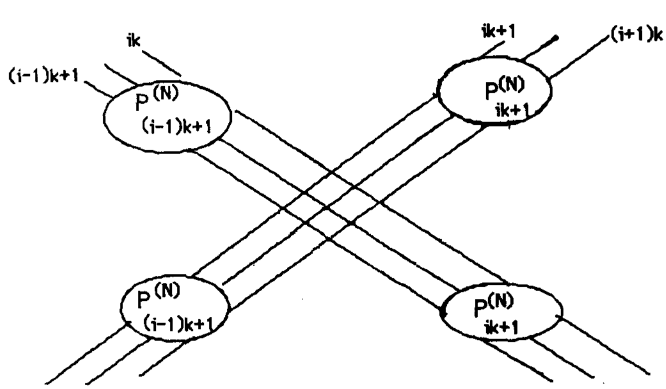

Fig. 2. Generator G_i of $B_n^{[s]}$. Note that $k = N-1$.

Using (1.1) and (1.7), it can be shown that the generators $G_1, G_2, \ldots, G_{n-1}$ are the defining relations of the braid group

$$G_iG_k = G_kG_i, \qquad |i-k| \geq 2, \qquad G_iG_{i+1}G_i = G_{i+1}G_iG_{i+1}$$
$$(G_i - c_1)(G_i - c_2)\ldots(G_i - c_N) = 0 \tag{2.9}$$

where, for $r = 1, 2, \ldots, N$,

$$c_r = (-1)^{N+r} t^{N(N-1)/2 - r(r-1)/2} \tag{2.10}$$

$N=3$. Let us now consider the case $N=3$. From equation (2.10) defining c_r we have

$$c_1 = t^3, \qquad c_2 = -t^2, \qquad c_3 = 1$$

and equation (2.10) becomes

$$G_i^3 = aG_i^2 + bG_i + c \tag{2.11}$$

where

$$a = 1 - t^2 + t^3, \qquad b = t^2 - t^3 + t^3, \qquad c = -t^5$$

Our objective is to derive the recursive formula for expressing G_i^n in terms of G_i and G_i^{-1}.

First we consider the case $n > 0$. We can write G_i as

$$G_i = \mathbf{M}_1(t, 1)G_i + \mathbf{M}_2(t, 1) + \mathbf{M}_3(t, 1)G_i^{-1}$$

where

$$\mathbf{M}_1(t, 1) = 1, \qquad \mathbf{M}_2(t, 1) = 0, \qquad \mathbf{M}_3(t, 1) = 0 \tag{2.12}$$

Now, if we multiply both sides of equation (2.11) by G_i^{-1}, we obtain

$$G_i^2 = aG_i + b + cG_i^{-1} \tag{2.13}$$

which can be written as

$$G_i^2 = \mathbf{M}_1(t, 2)G_i + \mathbf{M}_2(t, 2) + \mathbf{M}_3(t, 2)G_i^{-1}$$

where

$$\mathbf{M}_1(t, 1) = a, \qquad \mathbf{M}_2(t, 1) = b, \qquad \mathbf{M}_3(t, 1) = c$$

Applying G_i on both sides of equation (2.13), we get

$$G_i^3 = \mathbf{M}_1(t, 2)G_i^2 + \mathbf{M}_2(t, 2)G_i + \mathbf{M}_3(t, 2) \tag{2.15}$$

By substituting G_i^2 from equation (2.13), equation (2.15) becomes

$$G_i^3 = [a\mathbf{M}_1(t, 2) + \mathbf{M}_2(t, 2)]G_i + [b\mathbf{M}_1(t, 2) + \mathbf{M}_3(t, 2)] + c\mathbf{M}_1(t, 2)G_i^{-1}$$

or

$$G_i^3 = \mathbf{M}_1(t, 3)G_i + \mathbf{M}_2(t, 3) + \mathbf{M}_3(t, 3)G_i^{-1}$$

where

$$\mathbf{M}_1(t, 3) = a\mathbf{M}_1(t, 2) + \mathbf{M}_2(t, 2)$$
$$\mathbf{M}_2(t, 3) = b\mathbf{M}_1(t, 2) + \mathbf{M}_3(t, 2)$$
$$\mathbf{M}_3(t, 3) = c\mathbf{M}_1(t, 2)$$

Now, it is straightforward to generalize for any power $n > 0$,

$$G_i^n = \mathbf{M}_1(t, n)G_i + \mathbf{M}_2(t, n) + \mathbf{M}_3(t, n)G_i^{-1} \tag{2.16}$$

where

$$\begin{aligned} \mathbf{M}_1(t, n) &= a\mathbf{M}_1(t, n-1) + \mathbf{M}_2(t, n-1) \\ \mathbf{M}_2(t, n) &= b\mathbf{M}_1(t, n-1) + \mathbf{M}_3(t, n-1) \\ \mathbf{M}_3(t, n) &= c\mathbf{M}_1(t, n-1) \end{aligned} \tag{2.17}$$

For $n < 0$ we proceed similarly and get

$$G_i^{-p} = [\mathbf{V}_1(t, p)G_i + \mathbf{V}_2(t, p) + \mathbf{V}_3(t, p)G_i^{-1}]/c^{p-1} \tag{2.18}$$

and the functions $\mathbf{V}_1(t, p)$, $\mathbf{V}_2(t, p)$, and $\mathbf{V}_3(t, p)$ are generated from the following recurrence relations:

$$\mathbf{V}_1(t, 1) = 0; \qquad \mathbf{V}_2(t, 1) = 0; \qquad \mathbf{V}_3(t, 1) = 1$$

and, for $p \geq 2$,

$$\mathbf{V}_1(t, p) = \mathbf{V}_3(t, p-1)$$
$$\mathbf{V}_2(t, p) = c\mathbf{V}_1(t, p-1) - a\mathbf{V}_3(t, p-1)$$
$$\mathbf{V}_3(t, p) = c\mathbf{V}_2(t, p-1) - b\mathbf{V}_3(t, p-1)$$

To complete our setup, it is easy to prove the following relations, which are valid for any braid group:

$$G_{i+1}^n G_i G_{i+1} = G_i G_{i+1} G_i^n \tag{2.19}$$

$$G_{i+1}^n G_i^{-1} G_{i+1}^{-1} = G_i^{-1} G_{i+1}^{-1} G_i^n \tag{2.20}$$

3. EXAMPLES

Example 3.1

Consider the trefoil knot b_1^3. From equations (2.16) and (2.17) for $n = 3$ and $i = 1$ we have

$$G_1^3 = \mathbf{M}_1(t, 3)G_1 + \mathbf{M}_2(t, 3) + \mathbf{M}_3(t, 3)G_1^{-1} \tag{3.1a}$$

where

$$\mathbf{M}_1(t,3)=1-t^2+t^3+t^4-t^5+t^6 \tag{3.1b}$$

$$\mathbf{M}_2(t,3)=t^2-t^3-t^4+2t^5-t^6-t^7+t^8 \tag{3.1c}$$

$$\mathbf{M}_3(t,3)=-t^5+t^7-t^8 \tag{3.1d}$$

The generalized Ocneanu trace function $\psi^{[s]}(\cdot)$ defined on $B_k^{[s]}$ is related to the Ocneanu trace function $\psi(\cdot)$ defined on $B_k^{[1/2]}$ through the equation

$$\psi^{[s]}(A)=\psi(A)/[\psi(P_j)]^k, \qquad A\in B_k^{[s]} \tag{3.2}$$

$\psi^{[s]}(\cdot)$ satisfies the normalization and Markov properties, namely

$$\psi^{[s]}(I)=1 \tag{3.3a}$$

$$\psi^{[s]}(AB)=\psi^{[s]}(BA), \qquad A,B\in B_k^{[s]} \tag{3.3b}$$

$$\psi[s](AG_k)=Z\psi^{[s]}(A), \qquad A,B\in B_{k+1}^{[s]} \tag{3.3c}$$

$$\psi^{[s]}(AG_k^{-1})=\bar{Z}\psi^{[s]}A, \qquad A,B\in B_{k+1}^{[s]} \tag{3.3d}$$

where

$$Z=\frac{(1-t)(1-t^2)\ldots(1-t^{N-1})}{(1-\omega t)(1-\omega t^2)\ldots(1-\omega t^{N-1})} \tag{3.4a}$$

$$\bar{Z}=\frac{\omega^{N-1}(1-t)(1-t^2)\ldots(1-t^{N-1})}{(1-\omega t)(1-\omega t^2)\ldots(1-\omega t^{N-1})} \tag{3.4b}$$

Further, the two-variable link polynomial $\alpha_\omega^{[s]}(\cdot)$ can be written as

$$\alpha_\omega^{[s]}(A)=(\bar{Z}Z)^{-(k-1)/2}(\bar{Z}/Z)^{e(A)/2}\psi^{[s]}(A), \qquad A\in B_k^{[s]} \tag{3.5}$$

where $e(A)$ is the exponent sum of A. From equations (3.1a), (3.2a), (3.2c), and (3.2d) we get

$$\psi^{[1]}(G_1^3)=\mathbf{M}_1(t,3)Z+\mathbf{M}_2(t,3)+\mathbf{M}_3(t,3)\bar{Z} \tag{3.6}$$

For $N=3$, equations (3.4a), (3.4b) become

$$Z=\frac{(1-t)(1-t^2)}{(1-\omega t)(1-\omega t^2)} \tag{3.7a}$$

$$\bar{Z}=\frac{\omega^2(1-t)(1-t^2)}{(1-\omega t)(1-\omega t^2)} \tag{3.7b}$$

Now, $e(G_1^3)=3$, and for $k=2$, equations (3.1b)-(3.1d), (3.5), (3.6), (3.7a), and (3.7b) lead to

$$\alpha_\omega^{[1]}(G_1^3)=\omega^2(1+t^3+t^4+t^6-t^3\omega-t^4\omega-t^6\omega-t^7\omega+t^7\omega^2)$$

Example 3.2

Consider the knot $b_1b_2^2b_1b_2^{-9}b_1b_2^9$. Let $A_1=G_1G_2^2G_1G_2^{-9}G_1G_2^9$, using equation (3.2b) and relation (2.19), we get

$$\begin{aligned}\psi^{[1]}(A_1)&=\psi^{[1]}(G_2^9G_1G_2G_2G_1G_2^{-9}G_1)\\&=\psi^{[1]}(G_1G_2G_1^9G_2G_1G_2^{-9}G_1)\\&=\psi^{[1]}(G_1^3G_2^2)\\&=[\mathbf{M}_1(t,3)Z+\mathbf{M}_2(t,3)+\mathbf{M}_3(t,3)\bar{Z}](aZ+b+c\bar{Z})\\&=[(-1+t)^2(1+t)(1+t^3+t^4+t^6-t^3\omega-t^4\omega-t^6\omega-t^7\omega+t^7\omega^2)\\&\quad\times(1-t-t^2+2t^3-t^5+t^6-t^3\omega+t^5\omega-t^6\omega-t^7\omega+t^7\omega^2)]\\&\quad\times[(-1+t\omega)^2(-1+t^2\omega)^2]^{-1}\end{aligned}$$

Using equation (3.5), $k=3$, $e(A_1)=5$, we get

$$\begin{aligned}\alpha_\omega^{[1]}(A_1)=\omega^3[1&+t^3+t^4+t^6-t^3\omega-t^4\omega-t^6\omega-t^7\omega+t^7\omega^2)\\&\times(1-t-t^2+2t^3-t^5+t^6-t^3\omega+t^5\omega-t^6\omega-t^7\omega+t^7\omega^2)^2]\\&\times[(-1+t)^2(1+t)]^{-1}\end{aligned}$$

Example 3.3

Consider the knot $\mathbf{7}_3=b_1b_2^2b_1^{-2}b_2^5$; let $A_2=G_1G_2^2G_1^{-2}G_2^5$. From equation (2.16) we get

$$\begin{aligned}A_2=\mathbf{M}_1(t,5)G_1G_2^2G_1^{-2}G_2&+\mathbf{M}_2(t,5)G_1G_2^2G_1^{-2}\\&+\mathbf{M}_3(t,5)G_1G_2^2G_1^{-2}G_2^{-1}\end{aligned}\tag{3.8}$$

Set $A_{21}=G_1G_2^2G_1^{-2}G_2$, $A_{22}=G_1G_2^2G_1^{-2}$, and $A_{23}=G_1G_2^2G_1^{-2}G_2^{-1}$. Using equation (3.2b) and relation (2.19), one can reduce $\psi^{[1]}(A_{21})$ as follows:

$$\begin{aligned}\psi^{[1]}(A_{21})&=\psi^{[1]}(G_2G_1G_2^2G_1^{-2})\\&=\psi^{[1]}(G_1^2G_2G_1G_1^{-2})\\&=\psi^{[1]}(G_2G_1)=Z^2\end{aligned}\tag{3.9a}$$

In the same fashion we can reduce $\psi^{[1]}(A_{22})$ and $\psi^{[1]}(A_{23})$, namely

$$\psi^{[1]}(A_{22}) = \psi^{[1]}(G_2^2 G_1^{-1}) = \bar{Z}(aZ + b + c\bar{Z}) \tag{3.9b}$$

$$\psi^{[1]}(A_{23}) = \psi^{[1]}(G_2^{-1} G_1 G_2 G_2 G_1^{-2}) = \psi^{[1]}(G_2 G_1^{-1} G_2 G_1^{-1}) \tag{3.9c}$$

Since for $N=3$

$$G_i = P_{2i-1} P_{2i+1} g_{2i} g_{2i-1} g_{2i+1} g_{2i} P_{2i-1} P_{2i+1} \tag{3.10}$$

it is easy to see that

$$G_i^{-1} = (-tG_i + \beta P_{2i-1} P_{2i+1} g_{2i} P_{2i-1} P_{2i+1} + t^4 P_{2i-1} P_{2i+1} - b)/c \tag{3.11}$$

where

$$\beta = t(1-t)(1+t)^2, \qquad P_i = (t+g_i)/(1+t)$$

From equations (3.2), (3.10), and (3.11) we have

$$\begin{aligned}\psi^{[1]}(A_{23}) = {} & [(-1+t)^4(1+t)^2(1 - \omega - t\omega + t^2\omega - t^4\omega + t\omega^2 - t^2\omega^2 + 3t^4\omega^2 \\ & - t^6\omega^2 + t^7\omega^2 - t^4\omega^3 + t^6\omega^3 - t^7\omega^3 - t^8\omega^3 + t^8\omega^4)] \\ & \times [t^4(-1+t\omega)^2(-1+t^2\omega)^2]^{-1}\end{aligned}$$

Substituting for $\mathbf{M}_1(t, 5)$, $\mathbf{M}_2(t, 5)$, and $\mathbf{M}_3(t, 5)$ in equation (3.8) and using equation (3.5), we get

$$\begin{aligned}\alpha^{[1]}(t, \omega) = \omega(1 & - t - t^2 + 2t^3 - 2t^5 + t^6 + t^7 - t^8 + t^9 - t^{11} + t^{12} + t\omega \\ & + t^2\omega - 2t^3\omega + 4t^5\omega - t^6\omega - 3t^7\omega + 3t^8\omega + t^9\omega - 2t^{10}\omega + 2t^{11}\omega \\ & - t^{13}\omega + t^{14}\omega - 2t^5\omega^2 + 3t^7\omega^2 - 2t^8\omega^2 - 3t^9\omega^2 + 4t^{10}\omega^2 \\ & - 2t^{12}\omega^2 + 2t^{13}\omega^2 - t^{15}\omega^2 + t^{16}\omega^2 - t^7\omega^3 + t^9\omega^3 - 2t^{10}\omega^3 \\ & - 2t^{11}\omega^3 + t^{12}\omega^3 - t^{13}\omega^3 - 2t^{14}\omega^3 - t^{16}\omega^3 - t^{17}\omega^3 + t^{11}\omega^4 \\ & + t^{14}\omega^4 + t^{15}\omega^4 + t^{17}\omega^4)\end{aligned}$$

APPENDIX

Birman Example

The two closed braids $A = (b_1 b_2 b_1)^4 b_1^{-12} b_2^6$ and $B = b_1^{-6} b_1^{12}$ cannot be distinguished by Jones polynomials, since for $N=2$ we find that $\alpha^{[1/2]}(t, \omega)$

for A and B is the same, namely

$$\alpha^{[1/2]}(t,\omega)=[\omega^2(-1+t-t^2+t^3-t^4+\omega-t\omega+t^2\omega-t^3\omega+t^4\omega-t^5\omega+t^6\omega)$$
$$\times(-1+t-t^2+t^3-t^4+t^5-t^6+t^7-t^8+t^9-t^{10}+t^{11}$$
$$-t^{12}+t^2\omega-t^3\omega+t^4\omega-t^5\omega+t^6\omega-t^7\omega+t^8\omega-t^9\omega+t^{10}\omega$$
$$-t^{11}\omega+t^{12}\omega)]$$
$$\times[(-1+t)^2t^5]^{-1}$$

On the other hand, for $N=3$, we find that $\alpha^{[1]}(t,\omega)$ of A is given by

$$\alpha^{[1]}(t,\omega)=[\omega^4(1-2t+5t^3-5t^4-5t^5+11t^6+t^7-13t^8+2t^9$$
$$+11t^{10}+2t^{11}-12t^{12}-10t^{13}+21t^{14}+14t^{15}-36t^{16}-6t^{17}+45t^{18}$$
$$-10t^{19}-37t^{20}+15t^{21}+24t^{22}-5t^{23}-20t^{24}-9t^{25}+29t^{26}$$
$$+13t^{27}-41t^{28}-t^{29}+40t^{30}-12t^{31}-28t^{32}+17t^{33}+13t^{34}-13t^{35}$$
$$-t^{36}+2t^{37}+5t^{39}-5t^{40}-3t^{41}+7t^{42}-2t^{43}-4t^{44}+4t^{45}-2t^{47}+t^{48}$$
$$-\omega+t\omega+2t^2\omega-6t^3\omega+2t^4\omega+11t^5\omega-14t^6\omega-8t^7\omega+26t^8\omega$$
$$-3t^9\omega-28t^{10}\omega+10t^{11}\omega+22t^{12}\omega-3t^{13}\omega-23t^{14}\omega-14t^{15}\omega$$
$$+42t^{16}\omega+22t^{17}\omega-73t^{18}\omega-5t^{19}\omega+90t^{20}\omega-26t^{21}\omega-76t^{22}\omega$$
$$+38t^{23}\omega+49t^{24}\omega-21t^{25}\omega-40t^{26}\omega-5t^{27}\omega+55t^{28}\omega$$
$$+12t^{29}\omega-77t^{30}\omega+6t^{31}\omega+79t^{32}\omega-31t^{33}\omega-57t^{34}\omega+42t^{35}\omega$$
$$+24t^{36}\omega-32t^{37}\omega+8t^{39}\omega-2t^{40}\omega+7t^{41}\omega-9t^{42}\omega-5t^{43}\omega$$
$$+14t^{44}\omega-4t^{45}\omega-9t^{46}\omega+8t^{47}\omega+t^{48}\omega-5t^{49}\omega+2t^{50}\omega+t^{51}\omega$$
$$-t^{52}\omega+t\omega^2-2t^2\omega^2+t^3\omega^2+4t^4\omega^2-7t^5\omega^2+t^6\omega^2+12t^7\omega^2$$
$$-15t^8\omega^2-5t^9\omega^2+27t^{10}\omega^2-13t^{11}\omega^2-23t^{12}\omega^2+24t^{13}\omega^2$$
$$+9t^{14}\omega^2-14t^{15}\omega^2-4t^{16}\omega^2-11t^{17}\omega^2+27t^{18}\omega^2+24t^{19}\omega^2$$
$$-69t^{20}\omega^2+91t^{22}\omega^2-42t^{23}\omega^2-71t^{24}\omega^2+62t^{25}\omega^2+32t^{26}\omega^2$$
$$-39t^{27}\omega^2-16t^{28}\omega^2+t^{29}\omega^2+39t^{30}\omega^2+14t^{31}\omega^2-72t^{32}\omega^2$$
$$+7t^{33}\omega^2+83t^{34}\omega^2-42t^{35}\omega^2-58t^{36}\omega^2+60t^{37}\omega^2+15t^{38}\omega^2$$
$$-43t^{39}\omega^2+13t^{40}\omega^2+9t^{41}\omega^2-9t^{42}\omega^2+12t^{43}\omega^2-9t^{44}\omega^2-8t^{45}\omega^2$$
$$+18t^{46}\omega^2-6t^{47}\omega^2-11t^{48}\omega^2+12t^{49}\omega^2-7t^{51}\omega^2+4t^{52}\omega^2+t^{53}\omega^2$$
$$-2t^{54}\omega^2+t^{55}\omega^2-t^4\omega^3+t^5\omega^3+2t^6\omega^3-5t^7\omega^3+t^8\omega^3+8t^9\omega^3$$
$$-10t^{10}\omega^3-3t^{11}\omega^3+17t^{12}\omega^3-9t^{13}\omega^3-13t^{14}\omega^3+15t^{15}\omega^3+2t^{16}\omega^3$$

$$
\begin{aligned}
&-6t^{17}\omega^3-12t^{19}\omega^3+17t^{20}\omega^3+21t^{21}\omega^3-47t^{22}\omega^3-6t^{23}\omega^3\\
&+64t^{24}\omega^3-23t^{25}\omega^3-53t^{26}\omega^3+36t^{27}\omega^3+29t^{28}\omega^3-23t^{29}\omega^3\\
&-18t^{30}\omega^3-3t^{31}\omega^3+31t^{32}\omega^3+14t^{33}\omega^3-55t^{34}\omega^3+63t^{36}\omega^3\\
&-25t^{37}\omega^3-47t^{38}\omega^3+38t^{39}\omega^3+18t^{40}\omega^3-28t^{41}\omega^3+t^{42}\omega^3\\
&+6t^{43}\omega^3-t^{44}\omega^3+7t^{45}\omega^3-9t^{46}\omega^3-5t^{47}\omega^3+14t^{48}\omega^3-4t^{49}\omega^3\\
&-9t^{50}\omega^3+8t^{51}\omega^3+t^{52}\omega^3-5t^{53}\omega^3+2t^{54}\omega^3+t^{55}\omega^3-t^{56}\omega^3+t^{8}\omega^4\\
&-2t^{9}\omega^4+4t^{11}\omega^4-4t^{12}\omega^4-2t^{13}\omega^4+7t^{14}\omega^4-3t^{15}\omega^4-5t^{16}\omega^4\\
&+5t^{17}\omega^4+t^{19}\omega^4-10t^{21}\omega^4+8t^{22}\omega^4+15t^{23}\omega^4-22t^{24}\omega^4\\
&-9t^{25}\omega^4+32t^{26}\omega^4-5t^{27}\omega^4-27t^{28}\omega^4+11t^{29}\omega^4+16t^{30}\omega^4\\
&-5t^{31}\omega^4-10t^{32}\omega^4-7t^{33}\omega^4+16t^{34}\omega^4+13t^{35}\omega^4-28t^{36}\omega^4\\
&-5t^{37}\omega^4+32t^{38}\omega^4-8t^{39}\omega^4-24t^{40}\omega^4+15t^{41}\omega^4+10t^{42}\omega^4\\
&-11t^{43}\omega^4+t^{45}\omega^4+5t^{47}\omega^4-5t^{48}\omega^4-3t^{49}\omega^4+7t^{50}\omega^4\\
&-2t^{51}\omega^4-4t^{52}\omega^4+4t^{53}\omega^4-2t^{55}\omega^4+t^{56}\omega^4)]\\
&\times[(-1+t)^4t^{14}(1+t)^2]^{-1}
\end{aligned}
$$

and $\alpha^{[1]}(t,\omega)$ for B is given by

$$
\begin{aligned}
\alpha^{[1]}(t,\omega)=\,&[\omega^4(1-t+t^3-t^4+t^5+t^6-2t^7+2t^9-t^{10}-t^{11}+t^{12}-\omega+t^2\omega\\
&-t^3\omega-t^6\omega+t^7\omega-2t^9\omega+t^{10}\omega+2t^{11}\omega-2t^{12}\omega-t^{13}\omega\\
&+2t^{14}\omega-t^{16}\omega+t\omega^2-t^2\omega^2+t^4\omega^2-t^5\omega^2+t^7\omega^2-t^{11}\omega^2+t^{12}\omega^2\\
&+t^{13}\omega^2-2t^{14}\omega^2+2t^{16}\omega^2-t^{17}\omega^2-t^{18}\omega^2+t^{19}\omega^2)\\
&\times(1-t-t^2+2t^3-2t^5+t^6+t^7-t^8+t^{12}-t^{13}-t^{14}+2t^{15}-2t^{17}\\
&+t^{18}+t^{19}-t^{20}+t^{24}-t^{26}+t^{27}-t^{29}+t^{30}-t^{32}+t^{33}-t^{35}+t^{36}-t^3\omega\\
&+2t^5\omega-t^6\omega-2t^7\omega+2t^8\omega+t^9\omega-2t^{10}\omega+t^{12}\omega-t^{15}\omega+2t^{17}\omega\\
&-t^{18}\omega-2t^{19}\omega+2t^{20}\omega+t^{21}\omega-2t^{22}\omega+t^{24}\omega-t^{25}\omega-t^{28}\omega+t^{29}\omega\\
&-t^{31}\omega+t^{32}\omega-t^{34}\omega+t^{35}\omega-t^{37}\omega+t^7\omega^2-t^8\omega^2-t^9\omega^2+2t^{10}\omega^2\\
&-2t^{12}\omega^2+t^{13}\omega^2+t^{14}\omega^2-t^{15}\omega^2+t^{19}\omega^2-t^{20}\omega^2-t^{21}\omega^2\\
&+2t^{22}\omega^2-2t^{24}\omega^2+t^{25}\omega^2+t^{26}\omega^2-t^{27}\omega^2+t^{28}\omega^2-t^{30}\omega^2\\
&+t^{31}\omega^2-t^{33}\omega^2+t^{34}\omega^2-t^{36}\omega^2+t^{37}\omega^2)]\\
&\times[(-1+t)^4t^{16}(1+t)^2]^{-1}
\end{aligned}
$$

They are clearly different.

ACKNOWLEDGMENT

I thank Dr. E. Ahmed for useful comments and discussions.

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